TORSION IN TILING HOMOLOGY AND COHOMOLOGY

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Abstract. The first author's recent unexpected discovery of torsion in the integral cohomology of the Tübingen Triangle Tiling has led to a re-evaluation of current descriptions of and calculational methods for the topological invariants associated with aperiodic tilings. The existence of torsion calls into question the previously assumed equivalence of cohomological and K-theoretic invariants as well as the supposed lack of torsion in the latter. In this paper we examine in detail the topological invariants of canonical projection tilings; we extend results of Forrest, Hunton and Kellendonk to give a full treatment of the torsion in the cohomology of such tilings in codimension at most 3, and present the additions and amendments needed to previous results and calculations in the literature. It is straightforward to give a complete treatment of the torsion components for tilings of codimension 1 and 2, but the case of codimension 3 is a good deal more complicated, and we illustrate our methods with the calculations of all four icosahedral tilings previously considered. Turning to the K-theoretic invariants, we show that cohomology and K-theory agree for all canonical projection tilings in (physical) dimension at most 3, thus proving the existence of torsion in, for example, the K-theory of the Tübingen Triangle Tiling. The question of the equivalence of cohomology and K-theory for tilings of higher dimensional euclidean space remains open.

1. Introduction

For large classes of aperiodic tilings various methods have been devised to compute topological invariants such as the Cech cohomology or topological K-theory of the continuous hull, the cohomology of the discrete tiling groupoid or the K-theory of various associated C^*-algebras. All these invariants are closely related, many essentially equivalent, although, as a consequence of the work below, the equivalence of the cohomological and K-theoretic invariants is now only *a priori* true for tilings in dimension at most 3.

For tilings generated by a primitive substitution, Kellendonk [K97] showed how to compute the top dimension cohomology via the groupoid C^*-algebra, while Anderson and Putnam (AP) [AP98] constructed the hull as an inverse limit of a sequence of finite CW-complexes, allowing the determination of the

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cohomology and K-theory as direct limits. These direct limits can be computed explicitly, although in more than one dimension the procedure quickly becomes tedious. For some of the simpler two-dimensional examples it can still be carried out with the aid of a computer [G04].

For canonical projection tilings, Forrest, Hunton and Kellendonk (FHK) [FHK02b] related the Cech cohomology and K-theory to certain group or dynamical homology theories. For codimension up to three, they provided explicit formulæ [FHK02a, FHK02b], expressing the (rational) ranks of the homology groups in terms of the data defining the tiling. These formulæ are straightforward to evaluate [GK00, FHK01], and provide an efficient means to compute the cohomology and other related topological invariants. Moreover, from an earlier result [FH99] it was concluded that the integral cohomology groups for canonical projection tilings would always be free, and so they would be completely determined by their ranks, hence equivalently by their rational ranks. However, this premise was not correct in the generality needed, and thus there is more to the determination of the cohomology of a canonical projection tiling than its rational rank. The main purpose of this article is to provide a proper treatment of the integral cohomology; equivalently, we provide a discussion of the torsion part of the integral tiling homology.

A third approach, appearing in a recent paper of Kalugin [Ka05], gives yet another method to determine the (rational) Cech cohomology of canonical projection tilings. Although not stated in this generality, his method can be applied in essentially the same situations as the FHK method [FHK02a, FHK02b].

There are important differences, however, between these three approaches. Canonical projection tilings are obtained by projection from a periodic structure in $(n+d)$-dimensional space to d-dimensional physical space. The complementary n-dimensional space is the so-called internal or perpendicular space. Whereas FHK work entirely with internal space, Kalugin's method is set up in the $(n+d)$-dimensional space (wrapped onto a torus), and the AP method works exclusively in physical space (and knows nothing about internal space).

The classes of primitive substitution tilings on the one hand, and of canonical projection tilings on the other hand, have a non-trivial overlap, and for tilings in the overlap all three methods can be used for the computation of topological invariants. Exploiting this as a consistency check, the AP method had been implemented and applied to a number of popular two-dimensional tilings [G04], whose cohomology had been computed previously by the FHK method [GK00]. Whereas in all cases considered, the free part of the cohomology agreed with the previous results, for the Tübingen Triangle Tiling (TTT) an additional torsion part $\mathbb{Z}_5^2$ was found with the AP method in the top-dimensional cohomology group, which is in contradiction with the assumed freeness of cohomology in [FHK02a, FHK02b]. The question of torsion therefore had to be reconsidered.

For the codimension two case, to which the TTT belongs, it is indeed easy to reconcile the FHK theory with the new result. The correct torsion is in fact present in the formulæ of [FHK02a], but instead of reading it off, its authors neglected it, assuming it must vanish by appealing to the result of [FH99], which apparently is not applicable in the required generality. So, if properly interpreted, the results obtained with both methods agree in all known cases, and the results of [FHK02a] only have to be complemented with an explicit expression for the torsion part of the cohomology.

For codimension three the situation is somewhat more involved. Though it is still possible to use essentially the methods of [FHK02b], extra care is needed in order to keep track of torsion. The essence of the problem is that in the top dimension torsion can arise in more than one way, and there are potential extension problems to consider; these problems disappear for lower dimensional cohomology groups. Under some extra conditions, however, the torsion can still be determined formulaically, and even if these conditions are not satisfied, it is still possible, in favourable cases, to solve the extension problems and so determine the torsion completely, but only by appealing further to the geometry of the explicit example under consideration. We shall see an example of this in Section 6.4, where we compute for the dual canonical D_6 tiling.

An alternative perspective that is of value for codimension three schemes (and in principle is equally applicable for higher codimensions) is to work with homology or cohomology over a field of characteristic p. Here the formalism and methods of [FHK02a, FHK02b], which essentially worked with homology over a field of characteristic zero, carries over directly. By these methods we can obtain precise expressions for the *rank,* though not the *order,* of the p-torsion in integral cohomology, without having to deal with any of the extension problems.

Although in the codimension three case the characterisation of torsion is not as straightforward as one might have hoped, this is nevertheless the first time that a detailed and transparent characterisation of torsion has been obtained for a rather large class of tilings. In previous work, torsion was either believed to be absent [FHK02a, FHK02b], or was present only implicitly, and had to be computed tediously on a case by case basis [AP98]. Nevertheless, we have as yet little to say about the geometrical interpretation of torsion for these tilings.

A further point worth emphasizing is that, for canonical projection tilings in dimension $d \leqslant 3$ with finitely generated cohomology, the K-theory is still isomorphic to the cohomology, whether or not there is torsion in the cohomology. Previous arguments had relied on there being no torsion in the cohomology. As a result we have specific examples in which the K-theory of the tiling (including the K-theory of the associated non-commutative C^*-algberas) has torsion.

We note that the question of whether there are tilings in higher dimensions $d \geqslant 4$ for which the cohomology and K-theory differ remains open.

With this note we also take the opportunity to clarify two further discrepancies between Kalugin's results [Ka05] and our previous work [GK00, FHK02b]. Kalugin finds that all generalized Penrose tilings have isomorphic cohomology. We entirely agree with this; the deviating results in [GK00] are due to an incorrect parametrisation of the generalized Penrose tilings. The discrepancy in the cohomology of the icosahedral Ammann-Kramer tiling is due to an incorrect "simplification" in [FHK02b], which is not valid in general. If this simplification is not made, both results agree. We present corrected formulæ in Section 5.

The arrangement of this article is as follows. The core of our approach to the cohomology and K-theory of canonical projection tilings is the evaluation of certain group homologies $H_*(\Gamma; C^n)$, and the technique as developed in [FHK02a, FHK02b] is to compute these via subsidiary homologies $H_*(\Gamma; C^r)$ for $r < n$. We briefly review this material, introducing our main notation and explaining the role of these various homology groups in Section 2. This section is a summary of the relevant ideas of [FHK02a, FHK02b], and the reader may refer to those articles for a more complete background. In Section 3 we present an overview of which parts of previous articles still stand, irrespective of potential torsion, and which need alteration in the light of torsion. In this section we also observe that many of the previous formulæ, which were based on calculations over the rationals, remain valid when worked (with care) over a field of positive characteristic, such as the field of p elements $\mathbb{F}_p$, and in so doing give a powerful tool for the computation of the rank of the p-torsion when this occurs; we see an example of such a method used in Section 6, where some explicit examples are computed. Section 4 provides the body of work, giving formulæ for the homology of a canonical projection tiling in the cases of codimension 1, 2 and 3, including formulæ for any torsion in these homology groups. In Section 5 we give the correction to the codimension 3 formula of [FHK02a] mentioned above. In Section 6 we work some examples and provide a complete table of all previously published cohomology calculations, now amended so as to include their torsion components, where they exist. Finally, in Section 7 we finish with some general results on where torsion can (or cannot) occur for an arbitrary canonical projection tiling, and prove our theorem relating the K-theory to the cohomology.

2. Preliminaries

We start by briefly summarising our notation and the main objects of computation for our discussion of torsion in the cohomology and K-theory of canonical projection tilings. We use, generally, the notation and ideas of

[FHK02a, FHK02b], and we refer the reader to those articles for motivation and a discussion of the background material which gives a rationale for the following assertions.

Our set-up is as follows. We are aiming to compute the topological invariants for a canonical projection tiling or pattern in $\mathbb{R}^d$ with an effective codimension of n. Our most basic object to compute is a group homology $H_*(\Gamma; C^n)$, where Γ is a free abelian group of rank $d+n$ which can be identified with a dense lattice of the internal space $\mathbb{R}^n$. The coefficients C^n can be seen as the $\mathbb{Z}\Gamma$ module generated by indicator functions on compact polytopes, so-called $\mathcal{C}$-topes [FHK02b], which are intersections of Γ-translates of the acceptance domain and can be most conveniently described with the help of singular spaces (see next subsection). We have the related coefficient groups C^r for $0 \leqslant r \leqslant n$ which fit into an exact sequence

$$0 \to C^n \to C^{n-1} \to \cdots \to C^0 \to \mathbb{Z} \to 0\,. \tag{2.1}$$

This sequence allows direct calculations for $H_*(\Gamma; C^n)$ at least for small n, as in [FHK02a], which we investigate in greater detail below for $n = 2, 3$, while for general n it gives rise to the spectral sequence of [FHK02b]

$$E^1_{r,s} = H_s(\Gamma; C^r) \Longrightarrow H_{r+s}(\Gamma; \mathbb{Z}) = \mathbb{Z}^{\binom{n+d}{r+s}}\,. \tag{2.2}$$

Whether the analysis of $H_*(\Gamma; C^n)$ via the sequence (2.1) is done by hand or via this spectral sequence, the principle in both cases is to get at these homology groups by computing the lower terms $H_*(\Gamma; C^r)$ for $r < n$. Just as the C^n have a geometric origin, so do the C^r: these are modules of functions on r-dimensional subsets (singular r-planes) of $\mathbb{R}^n$, essentially the r-dimensional facets of the $\mathcal{C}$-topes. A key observation, which makes this method a practical one, is that the homology groups $H_*(\Gamma; C^r)$ split as

$$H_*(\Gamma; C^r) = \bigoplus_{\Theta} H_*(\Gamma^\Theta; C^r_\Theta) \tag{2.3}$$

where C^r_Θ is an analogous module of functions on a representative of a Γ-orbit of singular r-planes, and Γ^Θ is the stabiliser subgroup of this object.

We shall concentrate entirely on the case when these homologies are finitely generated as groups. Necessary and sufficient conditions for this are given in [FHK02b], and in particular it is necessary that the number $\nu = \frac{(n+d)}{n}$ is an integer (and hence n must divide d). In this case, the ranks of the stabiliser groups of the singular r-planes are then forced to be $r\nu$, and an important consequence of this for us here is the following bound on the non-zero ranks of the homology groups:

$$H_k(\Gamma; C^r) = 0 \quad \text{unless} \quad 0 \leqslant k \leqslant r(\nu - 1). \tag{2.4}$$

This puts, for example, significant constraints on the non-zero terms in the spectral sequence (2.2).

Given a computation of the homology $H_*(\Gamma; C^n)$, there is a duality which tells us how to compute the Cech cohomology of the hull MP of the pattern by the relation

$$H^i(MP; \mathbb{Z}) = H_{d-i}(\Gamma; C^n)\,; \tag{2.5}$$

in particular, note that this cohomology is potentially non-zero only for $0 \leqslant i \leqslant d$, and moreover $H^0(MP; \mathbb{Z}) = \mathbb{Z}$.

The main aim of Section 4 is to give an analysis of the torsion in the homology $H_*(\Gamma; C^n)$ for low values of n, basically $n \leqslant 3$. This is done most easily by navigating the coefficient sequence (2.1) by hand, much as it was done in the computations carried out in [FHK02a]. However, for our general results in Section 7 it is best to appeal to the spectral sequence approach (2.2); this should be thought of as morally the same approach, but using a more sophisticated method of book-keeping to keep track of n long exact sequences and their interactions all at once. We record a couple of observations that will be useful later. A good general reference to the ideas of spectral sequences and their application can be found in [Mc01].

Firstly, we note that the differentials act $d^j \colon E^j_{r,s} \to E^j_{r-j,s+j-1}$. The constraints on where the the homology groups $H_*(\Gamma; C^r)$ are non-zero noted above will often tell us that these differentials are trivial.

Secondly, the convergence of this spectral sequence means that the limit groups $E^\infty_{r,s}$ give the composition quotients of a filtration of $H_{r+s}(\Gamma; \mathbb{Z})$. Of particular interest to us below will be the cases where there are only one or two non-zero terms $E^\infty_{r,s}$ for given value of $r+s$. In particular, for given t, if there is only one non-zero $E^\infty_{r,s}$ with $r+s=t$ then $E^\infty_{r,s} = H_t(\Gamma; \mathbb{Z})$, and if there are two such non-zero groups, say $E^\infty_{r_1,s_1}$ and $E^\infty_{r_2,s_2}$ with $r_1 < r_2$, then there is a short exact sequence

$$0 \to E^\infty_{r_1,s_2} \to H_t(\Gamma; \mathbb{Z}) \to E^\infty_{r_2,s_2} \to 0\,. \tag{2.6}$$

We finish our preliminary observations with a note about the deduction of the K-theory from the homology and cohomology calculations. This is via a second spectral sequence, a version we shall call A^{**}_* say, of the Atiyah-Hirzebruch spectral sequence (AHSS) computing the topological K-theory of MP (and thence for various non-commutative theories too)

$$A_2^{r,s} = H^r(MP; K^s) \Longrightarrow K^{r+s}(MP) \tag{2.7}$$

where we recall that $K^s = \mathbb{Z}$ if s is even, and is zero if s is odd. This sequence is further described in [FH99] and can also be considered as a simple case of the Kasparov spectral sequence.

2.1. **Notation for geometric data.** We record the detailed notation used below and in [FHK02a, FHK02b] to describe the $\mathbb{Z}\Gamma$ modules C^r and various associated objects used in our computations.

Given a canonical projection scheme, the basic data consists of a triple $(V, \Gamma, \mathcal{W})$. Here V is a euclidean space of dimension n (the internal space), Γ a free abelian group of rank $n+d$ (the super-lattice), which we identify with a specific dense lattice in V, and a finite family $\mathcal{W} = \{W_i\}_i$ of affine hyperplanes whose normals span V (the spans of the linear components of the boundary of the acceptance domain; the latter is assumed to be polytopal). Furthermore, $\mathcal{W}$ cannot be written as a union $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$ such that the normals of $\mathcal{W}_i$ span complementary spaces V_i.

The homology groups $H_*(\Gamma, C^r)$ are defined as homologies of dynamical systems derived from the data $(V, \Gamma, \mathcal{W})$. The coefficient groups C^r are the $\mathbb{Z}\Gamma$-modules generated by indicator functions on r-dimensional facets of $\mathcal{C}$-topes, which now can be described as compact polytopes in V whose boundary belongs to some union $\bigcup_{(W,x)\in A}(W + x)$, where A is a finite subset of $\mathcal{W} \times \Gamma$. The homology groups depend therefore on the geometry and combinatorics of the intersections $\bigcap_{(W,x)\in A}(W + x)$, where A is some finite subset of $\mathcal{W} \times \Gamma$. We call such an affine subspace a singular r-space if its dimension is r. Let $\mathcal{P}_r$ be the set of singular r-spaces and denote the orbit space under the action by translation $I_r := \mathcal{P}_r/\Gamma$. The stabilizer $\{x \in \Gamma | \hat{\Theta} + x = \hat{\Theta}\}$ of a singular r-space $\hat{\Theta}$ depends only on the orbit class $\Theta \in I_r$ of $\hat{\Theta}$ and we denote it Γ^Θ. Fix $\hat{\Theta} \in \mathcal{P}_k$ for $r < k < \dim V$ and let $\mathcal{P}_r^{\hat{\Theta}} := \{\hat{\Psi} \in \mathcal{P}_r | \hat{\Psi} \subset \hat{\Theta}\}$. Then Γ^Θ (Θ the orbit class of $\hat{\Theta}$) acts on $\mathcal{P}_r^{\hat{\Theta}}$ and we let $I_r^{\hat{\Theta}} = \mathcal{P}_r^{\hat{\Theta}}/\Gamma^\Theta$. We can naturally identify $I_r^{\hat{\Theta}}$ with $I_r^{\hat{\Theta}'}$ if $\hat{\Theta}$ and $\hat{\Theta}'$ belong to the same Γ-orbit and so we define I_r^Θ, for the class $\Theta \in I_k$. Then $I_r^\Theta \subset I_r$ consists of those orbits of singular r-spaces which have a representative that lies in a singular space of class Θ. Finally we use the notation $L_r = |I_r|$, $\quad L_r^\Theta = |I_r^\Theta|$, and C_Θ^r for the submodule of C^r generated by indicator functions on r-dimensional facets lying in a singular r-space representing Θ. Below, and as in [FHK02a, FHK02b], we shall use α to index codimension 1 singular spaces and θ to index codimension 2 singular spaces.

Theorem 2.1 ([FHK02b]). *A necessary and sufficient condition for the homology groups $H_*(\Gamma; C^n)$ of a canonical projection tiling to be finitely generated is that the number L_0 is finite.*

3. What's right and what's wrong from before

The origins of this article lie with the discovery [G04] of 5-torsion in the homology of the Tübingen Triangle Tiling (TTT), a property that had been thought to be not possible in earlier work, including [FHK02a, FHK02b], and

hence had erroneously been overlooked in previous work. Torsion phenomena of various orders have now been discovered in other examples, and we give more details in Section 6.

The oversight of torsion in [FHK02a, FHK02b] is not due to errors *per se* in those articles, but rather to their use of the claim in [FH99] Proposition 3.2 that for $(X, \mathbb{Z}^d)$ a minimal Cantor dynamical system, there was no torsion in the homology groups $H_*(\mathbb{Z}^d; C(X, \mathbb{Z}))$; the TTT example shows that this result cannot hold in the generality needed for projection tiling systems. The lack of torsion was also used in previous work to deduce the K-theory of the hull; the presence of torsion means that the computation of the K-theory from the tiling homology groups is a lot more subtle than it would have been without torsion.

The existence of torsion thus requires discussion on two fronts: the elaboration of the work of [FHK02a, FHK02b] so as to include the overlooked torsion, and a more detailed examination of the resulting K-theory of the hull. In this section we outline which aspects of the calculations in print stand as they are, and which need modification in the light of potential torsion.

3.1. **Torsion-free part of tiling homology.** The main theoretical work on the homology associated to canonical projection patterns is to be found in [FHK02a, FHK02b], and explicit applications to compute particular examples have appeared in [FHK01, GK00]. The corrections needed to [FHK02a, FHK02b] concern only the explicit formulæ for $H_*(\Gamma; C^n)$ of [FHK02a] Theorem 64 and [FHK02b] Theorems 2.5, 2.6 and 2.7. We note that the calculations of any codimension 1 system remain correct, as there can be no torsion for such systems; we give details in Section 4. This remark includes all the work of [FHK02b] Chapter III. The statements about the Euler characteristic in [FHK02b] Chapter V (in particular Theorem 2.8) also remains correct: the Euler characteristic is essentially a rational invariant and so does not see torsion.

The strategy of deriving the formulæ in [FHK02a, FHK02b] for higher codimension systems basically yields the torsion free part of the homology, as it essentially works by deriving the rationalised homology groups $H_*(\Gamma; C^n) \otimes \mathbb{Q} = H_*(\Gamma; C^n \otimes \mathbb{Q})$. Indeed in [FHK02b] page 94 we define the numbers D_k as the ranks of these rational vector spaces, and the computions are correct as they stand with this interpretation of D_k: the problem is with deducing that once one has computed the rank of the free part, then one has the whole homology group computed. In the paper [FHK02a] the proof of Theorem 64 is correct up to the final line where the claim that the homology groups are torsion free enters for the first time. In particular, the formula

$$H_k(\Gamma; C^2) \cong H_{k+1}(\Gamma; C_0^0)/\mathrm{im}\beta_{k+1} \oplus H_k(\Gamma; C^1) \cap \ker\beta_k$$

is correct as it stands and we shall generalise this in Section 4. It is not correct, however, to assume that the quotient in this expression, $H_{k+1}(\Gamma; C_0^0)/\mathrm{im}\beta_{k+1}$, is torsion free, and in fact the TTT explicitly shows that there can be torsion in this term.

The consequence of this for the published calculations such as [FHK01, GK00] is that, apart from a computational error which we correct in Section 5, the ranks stated there are correct as regards the torsion free part of the homology, but there may be torsion terms that need to be added to give a complete computation for the integral theory. We note also that as all these calculations are for patterns with finitely generated homology, any potential torsion will necessarily be a direct summand. Explicitly, with the above interpretation of D_k we have

$$H_k(\Gamma; C^n) = \mathbb{Z}^{D_k} \oplus \tau_k \tag{3.1}$$

where each τ_k is a finite abelian group. One of the objectives of Section 4 may be seen as giving explicit descriptions of the groups τ_k.

3.2. **Homology over finite fields.** In [FHK02a, FHK02b], the assumption of torsion-freedom allowed one to effectively work with coefficients in a field, and use linear algebra; in the absence of torsion, it suffices to compute ranks of free abelian groups, which are equivalent to the dimensions of the corresponding vector spaces $H_*(\Gamma; C^n \otimes \mathbb{F})$, where $\mathbb{F}$ is any field (in the definition of D_k this was taken to be $\mathbb{Q}$, but if there is no torsion it could just as well be $\mathbb{F}_p$, the field with p elements for some prime p).

In the presence of torsion, while the linear algebra fails to compute the full integral homology, it is no longer the case that the choice of field $\mathbb{F}$ is irrelevant, and this insight allows us to use the formulæ of [FHK02b] to get a hold on the torsion. We first introduce some notation. For A a finitely generated abelian group and p any prime, the p-torsion of A is the subgroup of those elements of A which are annihilated by multiplication by some power of p. We continue to write D_k for the $\mathbb{Q}$ rank of $H_*(\Gamma; C^n \otimes \mathbb{Q})$. Let us also write D_k^p for the $\mathbb{F}_p$-rank of $H_*(\Gamma; C^n \otimes \mathbb{F}_p)$, and T_k^p for the rank of the p-torsion in $H_*(\Gamma; C^n)$; with the notation of equation (3.1) this means $T_k^p = \dim_{\mathbb{F}_p}(\tau_k \otimes \mathbb{F}_p)$. Interpret T_{-1}^p as zero. We make two observations.

First of all, the numbers D_k^p for low codimensions are computable from the formulæ in [FHK02b]. In fact, the formulæ given in the Theorems 2.6 and 2.7 hold exactly for computations of D_k^p, provided the ranks r_k and R_k are interpreted correctly: these numbers are defined in terms of the ranks of certain sublattices of exterior powers $\Lambda_r\Gamma$ and should be interpreted as the ranks of the same objects after tensoring with the field $\mathbb{F}_p$. In the presence of torsion these ranks depend on the value of the prime p considered; for example,

the rank of the submodule $5\mathbb{Z}$ in $\mathbb{Z}$ will be 1 after tensoring with $\mathbb{F}_p$ or $\mathbb{Q}$, unless $p = 5$, when the rank will be 0.

Secondly, the standard machinary of homological algebra gives us explicit relations ('universal coefficient formulæ' [Br]) between the homology groups $H_*(\Gamma; C^n)$ and groups $H_*(\Gamma; C^n \otimes R)$ for any commutative ring R. In our notation these yield the following result.

Proposition 3.1. *Assume that the homology groups $H_*(\Gamma; C^n)$ are finitely generated. Then*

$$D_k^p = D_k + T_k^p + T_{k-1}^p \,.$$

In particular, $H_*(\Gamma; C^n)$ is free of p-torsion if and only if $D_k = D_k^p$ for all $0 \leqslant k \leqslant d$.

Note that the Euler characteristic $\sum_k (-1)^k D_k = \sum_k (-1)^k D_k^p$ is independent of the field $\mathbb{F}$ chosen.

Remark 3.2. While this proposition gives a strong and computable hold on the torsion in $H_*(\Gamma; C^n)$, it will not tell the whole story in general. Basically, this result tells us only of the p-rank, and not of the order of p-torsion; for example, if it tells us that $T_k^p = 1$ we cannot deduce whether the p-part of τ_k is $\mathbb{Z}_p$ or $\mathbb{Z}_{p^s}$ for some other positive integer s. To handle this we develop explicit formulæ for the groups τ_k in the case of low codimension patterns in Section 4. Nevertheless, the same idea can be generalised further by considering homology over a more general ring R and it turns out that this also gives us a useful computational tool. For example, although working over $\mathbb{F}_2$ will not distinguish between $\mathbb{Z}_2$ and $\mathbb{Z}_4$ in the homology of the tiling, working over $R = \mathbb{Z}_4$ will. The disadvantage of this is that for rings R which are not fields, there is no simple analogue of the formulæ of [FHK02b], and computations must be hand-crafted to the specific case under consideration. We shall see an explicit example of this method in Section 6 when we compute the homology of the dual canonical D_6 tiling.

3.3. **Consequences of torsion for K-theory.** As noted before, one role of the supposed lack of torsion in tiling homology was its use in the deduction of the behaviour of the AHSS computing $K^*(MP)$. With torsion in $H_*(\Gamma; C^n)$ there is the potential of non-trivial differentials in A_*^{**}, and so a failure of the claim that the K-theory of MP is the direct sum of homology groups. In fact, things are not as bad as they might be, and we shall show in Section 7, that for canonical projection patterns in $\mathbb{R}^d$ with $d \leqslant 3$ and finitely generated homology, there are isomorphisms

$$K^0(MP) \cong \bigoplus_r H^{2r}(\Gamma; C^n) \qquad K^1(MP) \cong \bigoplus_r H^{2r+1}(\Gamma; C^n) \,. \tag{3.2}$$

Similar results for higher values of d also hold when there is no p-torsion for small primes p, and such results hold for all d under the assumptions that the homology groups $H_*(\Gamma; C^n)$ are finitely generated and torsion free.

4. Low codimension formulæ

In this section we produce specific descriptions for the homology of canonical projection tilings of codimension 1, 2 and 3, including as far as possible explicit expressions for torsion components in those degrees where the homology is potentially non-free. The codimension 3 case is complemented by the formulæ of Section 5, replacing those of Theorem 2.7 of [FHK02b].

4.1. **No torsion for Codimension 1.** Here we prove the assertion of the title, that for a codimension 1 (that is, $n = 1$) projection pattern, $H_k(\Gamma; C^1)$ has no torsion. In fact, this follows also from the analysis of the hull MP for an arbitrary codimension 1 pattern in Chapter III of [FHK02b], where it is shown that MP can be modelled by a punctured torus, but the argument below will demonstrate the spirit of the method used for higher codimensions. We shall work in greater generality than just the canonical patterns, and assume that there are potentially a number of distinct Γ-orbits of 'cut points' in the internal space; as before, we shall also call this number L_0. See [FHK02b] Chapter III for further discussion of the projection scheme.

Proof. As in Section 2, we compute the groups $H_*(\Gamma; C^1)$ where there is a short exact sequence of $\mathbb{Z}\Gamma$-modules

$$0 \longrightarrow C^1 \longrightarrow C^0 \longrightarrow \mathbb{Z} \longrightarrow 0\,.$$

In this sequence, $\mathbb{Z}$ carries the trivial Γ action, while the action of Γ on C^0 is free. This sequence gives rise to a long exact sequence in homology

$$\cdots \to H_{k+1}(\Gamma; \mathbb{Z}) \to H_k(\Gamma; C^1) \to H_k(\Gamma; C^0) \to \cdots\,.$$

Now $H_k(\Gamma; \mathbb{Z}) \cong \Lambda_k\Gamma$ is just the homology of a $(d+1)$-torus, so $H_k(\Gamma; \mathbb{Z})$ is free abelian of rank $\binom{d+1}{k}$. Meanwhile, the freeness of Γ on C^0 means that the homology groups $H_k(\Gamma; C^0)$ are zero for $k > 0$ and $H_0(\Gamma; C^0) = \mathbb{Z}^{L_0}$.

Our long exact sequence now tells us in dimensions $k > 0$ that $H_k(\Gamma; C^1) \cong \Lambda_{k+1}\Gamma \cong \mathbb{Z}^{\binom{d+1}{k+1}}$ and for dimension 0 there is an exact sequence

$$0 \to \Lambda_1\Gamma \to H_0(\Gamma; C^1) \to \mathbb{Z}^{L_0} \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where ϵ, which takes the sum over the coefficients in $\mathbb{Z}^{L_0}$, is onto. Hence $H_0(\Gamma; C^1) \cong \Lambda_1\Gamma \oplus \ker\epsilon$, and so in particular it is free abelian of rank $L_0 + d$. $\square$

4.2. **Codimension 2.** The codimension 2 theory follows pretty much the same analysis, via a pair of exact sequences, as in the proof of Theorem 64 in [FHK02a]. The exact sequence of coefficients 2.1 breaks into two short exact sequences

$$0 \to C^2 \to C^1 \to C^0_0 \to 0 \qquad 0 \to C^0_0 \to C^0 \to \mathbb{Z} \to 0\,. \tag{4.1}$$

The right hand sequence behaves identically to the calculations above for codimension 1 and we obtain

$$H_k(\Gamma; C^0_0) \cong \begin{cases} \mathbb{Z}^{\binom{d+2}{k+1}} = \Lambda_{k+1}\Gamma & \text{for} \quad k > 0, \\ \mathbb{Z}^{d+L_0+1} = \Lambda_1\Gamma \oplus \ker\epsilon & \text{for} \quad k = 0 \end{cases}$$

A similar calculation based on the exact sequences

$$0 \to C^1_\alpha \to C^0_\alpha \to \mathbb{Z} \to 0 \tag{4.2}$$

gives

$$H_k(\Gamma; C^1) \cong \begin{cases} \bigoplus_{\alpha\in I_1} \Lambda_{k+1}\Gamma^\alpha & \text{for} \quad k > 0, \\ \bigoplus_{\alpha\in I_1} (\Lambda_1\Gamma^\alpha \oplus \ker\epsilon^\alpha) & \text{for} \quad k = 0 \end{cases}$$

where Γ^α denotes the stabiliser subgroup of a cut line represented by α and $\epsilon^\alpha : H_0(\Gamma^\alpha; C^0_\alpha) \cong \mathbb{Z}^{L^\alpha_0} \to \mathbb{Z}$. We note by [FHK02b] page 97 that $H_*(\Gamma; C^2)$ is finitely generated if and only if L_0 is finite, and in this case $\nu = \frac{d+2}{2}$ is an integer and the rank of each Γ^α is ν. The number L^α_0 is the number of Γ^α orbits of cut points on a cut line represented by α.

Denote by β_k the homomorphism in $H_k(\Gamma; -)$ induced by the module homomorphism $C^1 \to C^0_0$. For dimensions $k > 0$ this identifies with the homomorphism

$$\bigoplus_{\alpha\in I_1} \Lambda_{k+1}\Gamma^\alpha \to \Lambda_{k+1}\Gamma$$

induced by the inclusions $\Gamma^\alpha \to \Gamma$. Similarly, β_0 identifies with the homomorphism

$$\bigoplus_{\alpha\in I_1} (\Lambda_1\Gamma^\alpha \oplus \ker\epsilon^\alpha) \to \Lambda_1\Gamma \oplus \ker\epsilon$$

induced by the inclusions $\Gamma^\alpha \to \Gamma$ and $C^0_\alpha \subset C^0$. The left hand sequence in (4.1) now gives a long exact sequence in homology, which at $H_k(\Gamma; C^2)$ may be written as the short exact sequence

$$0 \to \text{coker } \beta_{k+1} \to H_k(\Gamma; C^2) \to \ker \beta_k \to 0\,.$$

This sequence splits as ker β_k is a subgroup of a free abelian group. We obtain

Theorem 4.1. *Assuming we work with a canonical projection pattern with codimension 2 and finitely generated homology, then*

$$H_k(\Gamma; C^2) = \text{coker}\beta_{k+1} \oplus \ker\beta_k$$

and so τ_k, the torsion part of $H_k(\Gamma; C^2)$, is given by the torsion part of the cokernel of

$$\beta_{k+1} \colon \bigoplus_{\alpha \in I_1} \Lambda_{k+2}\Gamma^\alpha \to \Lambda_{k+2}\Gamma \,.$$

In particular, $H_k(\Gamma; C^2)$ is torsion free for $k \geqslant d/2$.

Proof. We have justified all but the final sentence. We have observed that the torsion in $H_k(\Gamma; C^2)$ is equivalent to the torsion in coker β_{k+1}. Clearly, there will be no torsion in this cokernel if each $\Lambda_{k+2}\Gamma^\alpha$ is zero, and this will be the case if $k+2$ exceeds the rank of Γ^α. Thus there is no torsion in $H_k(\Gamma; C^2)$ if $k+1 \geqslant \frac{d+2}{2}$, from which the inequality follows. $\square$

This final result, putting bounds on where torsion may appear, will be seen to be a special case of a result for arbitrary codimension in Section 7. Examples suggest that these bounds are best possible. For now we note that for canonical projection patterns with $n = d = 2$ and finitely generated homology (equivalently, L_0 finite), torsion can only appear in $H_0(\Gamma; C^2)$.

4.3. **Codimension 3.** The codimension 3 case is a good deal more complex than the codimension 2 theory, though the principles of computation remain the same. We consider here only the physically interesting case $d = 3$ for simplicity, though the method can be repeated to cover other codimension 3 cases. Note that the condition for finitely generated homology (that $\nu = \frac{n+d}{n}$ be an integer) means that the only canonical projection patterns with $d = 3$ and with finitely generated homology will be of codimension 1 or 3.

Computation proceeds as before. We could compute via the spectral sequence of [FHK02b], and in a sense this is what we do, but the need to describe the torsion terms via explicit homomorphisms, kernels and cokernels means that we must examine the details of the underlying exact couple, which amounts to saying that we compute the long exact sequences associated to the tower 2.1, which here breaks into the three short exact sequences

$$\begin{aligned} 0 \to C^0_0 \to C^0 \to \mathbb{Z} \to 0 \,, \\ 0 \to C^1_0 \to C^1 \to C^0_0 \to 0 \,, \\ 0 \to C^3 \to C^2 \to C^1_0 \to 0 \,. \end{aligned}$$

The dimension 3 internal space now has families of singular lines and singular planes. As in [FHK02b], and below in Theorem 5.1, we shall index by θ the lines and by α the planes. The rank of the main group Γ is 6. Finite generation of homology implies that the rank of the stabiliser Γ^α of a singular plane is 4, while the stabiliser Γ^θ of a singular line has rank 2.

The third of the short exact sequences above gives us a long exact sequence computing $H_*(\Gamma; C^3)$ with sections

$$\cdots \to H_s(\Gamma; C^3) \to H_s(\Gamma; C^2) \xrightarrow{\phi_s} H_s(\Gamma; C^1_0) \to H_{s-1}(\Gamma; C^3) \to \cdots \tag{4.3}$$

telling us we have short exact sequences

$$0 \to \text{coker } \phi_{s+1} \to H_s(\Gamma; C^3) \to \ker \phi_s \to 0\,. \tag{4.4}$$

In order to describe these kernel and cokernel terms, we must first describe the homologies $H_*(\Gamma; C^1_0)$ and $H_*(\Gamma; C^2)$. Computations for each are essentially identical to those for a codimension 2 system, as presented in the previous subsection. In particular, using the exact sequence of coefficients

$$0 \to C^1_0 \to C^1 \to C^0 \to \mathbb{Z} \to 0$$

we compute

$$H_s(\Gamma; C^1_0) = \begin{cases} 0 & \text{for} \quad s \geqslant 5, \\ \Lambda_{s+2}\Gamma & \text{for} \quad s \geqslant 2, \\ \Lambda_3\Gamma \oplus \ \ker \gamma_1 & \text{for} \quad s = 1, \\ \text{coker } \gamma_1 \oplus \ \ker \gamma_0 & \text{for} \quad s = 0, \end{cases} \tag{4.5}$$

where

$$\gamma_s \colon \bigoplus_{\theta \in I_1} \Lambda_{s+1}\Gamma^\theta \to \Lambda_{s+1}\Gamma \quad (s > 0), \qquad \gamma_0 \colon \bigoplus_{\theta \in I_1} \left(\Lambda_1\Gamma^\theta \oplus \ker \epsilon^\theta\right) \to \Lambda_1\Gamma \oplus \ker \epsilon$$

are both induced by the inclusions $\Gamma^\theta \to \Gamma$ and $C^0_\theta \subset C^0$. Here L_0 denotes as usual the number of Γ-orbits of cut points, and L^θ_0 denotes the number of Γ^θ-orbits of cut points on the line indexed by θ; finite generation of homology implies all these numbers to be finite [FHK02b]. Note that all the terms in (4.5) are free of torsion except possibly the coker γ_1 summand.

For each singular plane we have an analogous sequence

$$0 \to C^2_\alpha \to C^1_\alpha \to C^0_\alpha \to \mathbb{Z} \to 0$$

and the homology $H_*(\Gamma; C^2)$ splits as $\bigoplus_{\alpha \in I_2} H_*(\Gamma^\alpha; C^2_\alpha)$. We obtain

$$H_s(\Gamma^\alpha; C^2_\alpha) = \begin{cases} 0 & \text{for} \quad s \geqslant 3, \\ \Lambda_{s+2}\Gamma^\alpha = \mathbb{Z} & \text{for} \quad s = 2, \\ \Lambda_3\Gamma^\alpha \oplus \ \ker \beta^\alpha_1 & \text{for} \quad s = 1, \\ \text{coker } \beta^\alpha_1 \oplus \ \ker \beta^\alpha_0 & \text{for} \quad s = 0\,. \end{cases} \tag{4.6}$$

Here

$$\begin{aligned} \beta^\alpha_s \colon \quad & \textstyle\bigoplus_{\theta \in I^\alpha_1} \Lambda_{s+1}\Gamma^\theta \to \Lambda_{s+1}\Gamma^\alpha & \qquad (s > 0), \\ \beta^\alpha_0 \colon \quad & \textstyle\bigoplus_{\theta \in I^\alpha_1} \left(\Lambda_1\Gamma^\theta \oplus \ker \epsilon^\theta\right) \to \Lambda_1\Gamma^\alpha \oplus \ker \epsilon^\alpha. & \end{aligned}$$

The β^α_s are again induced by inclusion, and the only potential torsion term in (4.6) arises from the cokernel β^α_1 expression; all other terms are free abelian.

The expression for ϕ_s under the identifications (4.5, 4.6) can be obtained as in [FHK02b]: ϕ_s is a sum of morphisms $\phi^\alpha_s = (\tilde{\delta}^\alpha_{s2} \otimes 1)_*$, and the latter is

determined by the diagram

$$
\begin{array}{ccccccc}
\to & H_s(\Gamma; C^2_\alpha \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha]) & \to & \bigoplus_{\theta\in I_1^\alpha} \Lambda_{s+1}\Gamma^\theta & \xrightarrow{\beta_s^\alpha} & \Lambda_{s+1}\Gamma^\alpha & \to \\
 & \Big\downarrow \phi_s^\alpha & & \Big\downarrow j_s^\alpha & & \Big\downarrow \imath_s^\alpha & \\
\to & H_s(\Gamma; C^1_0) & \to & \bigoplus_{\theta\in I_1} \Lambda_{s+1}\Gamma^\theta & \xrightarrow{\gamma_s} & \Lambda_{s+1}\Gamma & \to
\end{array}
\tag{4.7}
$$

for $s > 0$, and
(4.8)

$$
\begin{array}{ccccccc}
\to H_0(\Gamma; C^2_\alpha \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha]) & \to & \bigoplus_{\theta\in I_1^\alpha} (\Lambda_1\Gamma^\theta \oplus \ker\epsilon^\theta) & \xrightarrow{\beta_0^\alpha} & \Lambda_1\Gamma^\alpha \oplus \ker\epsilon^\alpha & \to 0 \\
\Big\downarrow \phi_s^\alpha & & \Big\downarrow j_0^\alpha & & \Big\downarrow \imath_0^\alpha & \\
\to H_0(\Gamma; C^1_0) & \to & \bigoplus_{\theta\in I_1} (\Lambda_1\Gamma^\theta \oplus \ker\epsilon^\theta) & \xrightarrow{\gamma_0} & \Lambda_1\Gamma \oplus \ker\epsilon & \to 0
\end{array}
$$

where j_s^α and $\imath_s^\alpha$ are the obvious inclusions.

Lemma 4.2. *Consider a commutative diagram of abelian groups,*

$$
\begin{array}{ccccccc}
0 \to & A & \longrightarrow & B & \xrightarrow{f} & C & \to 0 \\
 & \downarrow \psi' & & \downarrow \psi & & \downarrow \psi'' & \\
0 \to & X & \longrightarrow & Y & \xrightarrow{g} & Z & \to 0,
\end{array}
\tag{4.9}
$$

in which the rows are short exact. If we assume that ψ'' is injective and C and Z are free abelian then $B \cong A \oplus C$, $Y \cong X \oplus Z$ and with respect to this decomposition $\psi = \psi' \oplus \psi''$.

Proof. Freeness of C and Z ensures that the exact rows split, and we can consider B and Y as direct sums as indicated. The issue is to show that with respect to such a decomposition the resulting matrix expression for ψ is diagonal. Simple diagram chasing shows that one off-diagonal term (the map from A to Z) will always be zero; thus ψ will have an expression

$$
\begin{pmatrix} \psi' & \zeta \\ 0 & \psi'' \end{pmatrix}
$$

for some homomorphism $\zeta\colon C \to X \subset Y$. If ζ is not the zero map, then there is an element $c \in C$ with $\psi(0, c) = \zeta(c)$ a non-zero element of the image of X in Y. Then

$$
0 = g\psi(0, c) = \psi''(f(0, c)) = \psi''(c)
$$

contradicting ψ'' being injective. $\square$

We apply this lemma to $\psi = \phi_s^\alpha$ to obtain

$$
\phi_s^\alpha \cong \tilde{i}^\alpha_{s+1} \oplus j_s^\alpha \big|_{\ker\beta_s^\alpha}
$$

where $\tilde{i}^\alpha_{s+1}$: coker$\beta^\alpha_{s+1} \to$ cokerγ_{s+1} is induced by i^α_{s+1}. The conditions are satisfied as $\ker\beta_s^\alpha$ and $\ker\gamma_s$ are free abelian and $j_s^\alpha\big|_{\ker\beta_s^\alpha}$ injective. Since ϕ_s

is the direct sum of the ϕ_s^α, we obtain the following description of the long exact sequence (only $s \leqslant 2$ is relevant for our considerations)

$$\begin{array}{ccccc}
\to H_2(\Gamma; C^3) \to & H_2(\Gamma; C^2) & \xrightarrow{\phi_2} & H_2(\Gamma; C^1_0) & \to \\
& \| & & \| & \\
& \bigoplus_{\alpha \in I_2} \Lambda_4 \Gamma^\alpha & \xrightarrow{\phi_2'} & \Lambda_4 \Gamma & \\
& & & & \\
\to H_1(\Gamma; C^3) \to & H_1(\Gamma; C^2) & \xrightarrow{\phi_1} & H_0(\Gamma; C^1_0) & \to \\
& \| & & \| & \\
& \bigoplus_{\alpha \in I_2} (\Lambda_3 \Gamma^\alpha \oplus \ker \beta_1^\alpha) & \xrightarrow{\phi_1' \oplus \phi_1''} & \Lambda_3 \Gamma \oplus \ker \gamma_1 & \\
& & & & \\
\to H_0(\Gamma; C^3) \to & H_0(\Gamma; C^2) & \xrightarrow{\phi_0} & H_0(\Gamma; C^1_0) & \to 0 \\
& \| & & \| & \\
& \bigoplus_{\alpha \in I_2} (\text{coker}\beta_1^\alpha \oplus \ker \beta_0^\alpha) & \xrightarrow{\phi_0' \oplus \phi_0''} & \text{coker}\gamma_1 \oplus \ker \gamma_0 &
\end{array} \tag{4.10}$$

where ϕ_s' and ϕ_s'' represent the homomorphisms $\tilde{i}^\alpha_{s+1}$ and $j^\alpha_s \big|_{\ker \beta^\alpha_s}$, respectively.

Theorem 4.3. *Consider a canonical projection pattern with $n = d = 3$ and assume the homology is finitely generated (equivalently* [FHK02b] *that L_0 is finite). Then*

$$H_s(\Gamma; C^3) = \begin{cases} 0 & \textit{for} \quad s \geqslant 4, \\ \mathbb{Z} & \textit{for} \quad s = 3, \\ \mathbb{Z}^6 \oplus \ker \left\{ \phi_2' \colon \bigoplus_{\alpha \in I_2} (\Lambda_4 \Gamma^\alpha) \to \Lambda_4 \Gamma \right\} & \textit{for} \quad s = 2, \end{cases}$$

and so there is no torsion in these degrees. There is potential torsion in degrees 0 and 1, and we have

$$\text{Torsion}\left(H_1(\Gamma; C^3) \right) = \text{Torsion}(\text{coker}\phi_2')$$

and an exact sequence (note the right hand map is not necessarily onto)

$$0 \to \text{Torsion}\left(\text{coker}\phi_1' \oplus \text{coker}\phi_1''\right) \to \text{Torsion}\left(H_0(\Gamma; C^3) \right) \to \text{Torsion}\left(\ker(\phi_0')\right).$$

Proof. These results can be read off from the long exact sequence (4.10) together with the observation that $\ker \phi_0''$ and $\ker \phi_1$ are torsion free. The last exact sequence arises, as taking the torsion is a left exact functor. $\square$

Note that the torsion of a codimension 3 tiling is determined by the above theorem without additional information only if $\ker \phi_0'$ is torsion free. Otherwise the exact sequence (4.4) for $s = 0$

$$0 \to \text{coker } \phi_1 \to H_0(\Gamma; C^3) \to \ker \phi_0 \to 0$$

presents an extension problem, whose solution requires further geometric input from the projection data. We see an example of such a problem in Section 6, where we compute the homology of the dual canonical D_6 tiling.

5. Corrected formulæ for the rational ranks of codimension 3 systems

We correct the formulæ of Theorem 2.7 of [FHK02b]. The formulæ given in [FHK02b] are correct only if the equation ([FHK02b], page 112, bottom half) $\langle \operatorname{im} j_s^\alpha \cap \ker \gamma_s : \alpha \in I_2 \rangle = \ker \gamma_s$ is valid. For the Ammann-Kramer tiling and the dual canonical D_6 tiling the rank of the left hand side is, however, one less than the rank of the right hand side.

The corrected formulæ for the ranks is obtained by replacing

$$\operatorname{rank} \ker \gamma_s = L_1 - \operatorname{rank}\langle \Lambda_{s+1}\Gamma^\Theta : \Theta \in I_1 \rangle$$

with

$$\operatorname{rank}\langle \operatorname{im} j_s^\alpha \cap \ker \gamma_s : \alpha \in I_2 \rangle = \operatorname{rank}\langle (\bigoplus_{\theta \in I_1^\alpha} \Lambda_{s+1}\Gamma^\theta) \cap \ker \gamma_s^\alpha : \alpha \in I_2 \rangle$$

where

$$\gamma_s^\alpha : \bigoplus_{\theta \in I_1^\alpha} \Lambda_{s+1}\Gamma^\theta \to \Lambda_{s+1}\Gamma$$

is the direct sum of the inclusions, i.e. $\gamma_s^\alpha(x_1, \cdots, x_{L_1^\alpha}) = x_1 + \cdots + x_{L_1^\alpha}$ (in the notation of [FHK02b] $\gamma_s^\alpha = \gamma_s \circ j_s^\alpha$.) This replacement is straightforward, yielding:

Theorem 5.1 (Erratum to Theorem 2.7 of [FHK02b]). *Given a projection method pattern with codimension* 3 *and finite* L_0 *then, for* $s > 0$,

$$\begin{aligned} D_s &= \binom{3\nu}{s+3} + L_2 \binom{2\nu}{s+2} + \sum_{\alpha \in I_2} L_1^\alpha \binom{\nu}{s+1} + L_1 \binom{\nu}{s+2} \\ &\quad - R_s - R_{s+1}, \\ D_0 &= \sum_{j=0}^{3} (-1)^j \binom{3\nu}{3-j} + L_2 \sum_{j=0}^{2} (-1)^j \binom{2\nu}{2-j} \\ &\quad + \sum_{\alpha \in I_2} L_1^\alpha \sum_{j=0}^{1} (-1)^j \binom{\nu}{1-j} + L_1 \sum_{j=0}^{2} (-1)^j \binom{\nu}{2-j} + e - R_1 \end{aligned}$$

where

$$\begin{aligned} R_s &= \operatorname{rank}\langle \Lambda_{s+2}\Gamma^\alpha : \alpha \in I_2 \rangle + \sum_{\alpha \in I_2} \operatorname{rank}\langle \Lambda_{s+1}\Gamma^\theta : \theta \in I_1^\alpha \rangle \\ &\quad + \operatorname{rank}\langle (\bigoplus_{\theta \in I_1^\alpha} \Lambda_{s+1}\Gamma^\theta) \cap \ker \gamma_s^\alpha : \alpha \in I_2 \rangle \end{aligned}$$

and the Euler characteristic is

$$e := \sum_s (-1)^s D_s = L_0 - \sum_{\alpha\in I_2} L_0^\alpha + \sum_{\alpha\in I_2}\sum_{\theta\in I_1^\alpha} L_0^\theta - \sum_{\theta\in I_1} L_0^\theta.$$

The above formulæ applied to the Ammann-Kramer tiling agree with the results of [Ka05]. The corrected rational ranks for the tilings of [FHK01] appear in Section 6.

6. Examples

6.1. **Remarks on the computations.** All examples discussed below have to some extent been calculated by computer. For this purpose, we have used the computer algebra system GAP [GAP], the GAP package Cryst [EGN97, Cryst], as well as further software written in the GAP language. It should be emphasized that these computations are not numerical, but use integers and rationals of arbitrary size or precision. Neglecting the possibility of programming errors, they must be regarded as exact.

One piece of information that needs to be computed is the set of all intersections of singular affine subspaces, along with their incidence relations. This is done with code based on the Wyckoff position routines from the Cryst package. The set of singular affine subspaces is invariant under the action of a space group. Cryst contains routines to compute intersections of such affine subspaces and provides an action of space group elements on affine subspaces, which allows to compute space group orbits. These routines, or variants thereof, are used to determine the space group orbits of representatives of the singular affine subspaces, and to decompose them into translation orbits. The intersections of the affine subspaces from two translation orbits is the union of finitely many translation orbits of other affine subspaces. These intersections can be determined essentially by solving a linear system of equations modulo lattice vectors, or modulo integers when working in a suitable basis. With these routines, it is possible to generate from a space group and a finite set $\mathcal{W}$ of representative singular affine spaces the set of all singular spaces, their intersections, and their incidences.

A further task is the computation of ranks, intersections, and quotients of free $\mathbb{Z}$-modules, and of homomorphisms between such modules, including their kernels and cokernels. These are standard algorithmic problems, which can be reduced to the computation of Smith and Hermite normal forms of integer matrices, including the necessary unimodular transformations [Coh95]. GAP already provides such routines, which are extensively used.

6.2. **The first appearance of torsion.** The first hint for the possible existence of torsion in tiling (co)homology was obtained [G04] by applying the AP

method [AP98] to several substitutional canonical projection tilings of dimension two. The purpose of this endeavour was to determine the action of the substitution on the tiling cohomology. We only sketch the general procedure here. In a first step, a large piece of the tiling is generated by substitution, and from the local tile neighborhoods a finite CW space Ω_1 is constructed, as detailed in [AP98]. The tiles are labelled according to their first corona, and each translation class of labelled tiles gives rise to a 2-cell in Ω_1. Boundaries of such 2-cells are identified whenever corresponding labelled tiles share such a boundary somewhere in the tiling. There is a natural action of the substitution on the (integral) co-chain groups of Ω_1, which induces a corresponding action on the cohomology groups $H^*(\Omega_1)$. The cohomology of the hull of the tiling is then obtained by taking the direct limit of $H^*(\Omega_1)$ under the iterated substitution action [AP98].

This procedure was applied to the Penrose tiling [deB81], the (undecorated) Ammann-Beenker tiling [Ben82, AGS92], and the Tübingen Triangle Tiling (TTT) [BKSZ90, KSB93]. Especially for the latter, it is a computational tour de force. For the TTT, Ω_1 turned out to have 860 2-cells, 1710 1-cells, and 880 0-cells. Computing the (integral) cohomology of such a large cell complex is computationally quite demanding, and it seems hardly possible to consider more complicated cases like the decorated versions of the Ammann-Beenker tiling [AGS92, Soc89] and the Socolar tiling [Soc89], let alone the three-dimensional examples. In all cases considered, all rational ranks of the cohomology groups agreed with the results obtained earlier by different methods [GK00]. The only difference was a torsion part $\mathbb{Z}_5^2$ in H^2 ($\cong H_0$) of the TTT, which according to [FHK02a, FHK02b] should not be there, and which did not go away even after an extensive review of the (rather complicated) computer program. This was the starting point of the revised theory of tiling (co)homology presented here.

6.3. **Codimension 2 examples with dihedral symmetry.** The codimension 2 examples discussed in the following have dihedral symmetry of order $2n$, with n even. The lattice Γ_n is given by the $\mathbb{Z}$-span of the vectors in the star $e_i = (\cos(\frac{2\pi i}{n}), \sin(\frac{2\pi i}{n}))$, $i = 0, \ldots, n-1$. The singular lines have special orientations with respect to this lattice. They are parallel to mirror lines of the dihedral group, which means that they are either *along* the basis vectors e_i, or *between* two neighboring basis vectors, i.e., along $e_i + e_{i+1}$. In all examples, one line from each translation orbit passes through the origin. We denote the sets of representative singular lines by $\mathcal{W}_n^a$ and $\mathcal{W}_n^b$, for lines along and between the basis vectors e_i. Denoting by V the internal space containing the singular lines, we can now describe the defining data of several well-known tilings, whose cohomology was already given in [GK00] (without torsion). The Penrose tiling [deB81] is defined by the triple $(V, \Gamma_{10}, \mathcal{W}_{10}^a)$, the Tübingen Triangle

Table 1. Homology of codimension 2 tilings with dihedral symmetry.

Tiling	H_0	H_1	H_2
Ammann-Beenker (undecorated)	$\mathbb{Z}^9$	$\mathbb{Z}^5$	$\mathbb{Z}^1$
Ammann-Beenker (decorated)	$\mathbb{Z}^{23}$	$\mathbb{Z}^9$	$\mathbb{Z}^1$
Penrose	$\mathbb{Z}^8$	$\mathbb{Z}^5$	$\mathbb{Z}^1$
generalized Penrose	$\mathbb{Z}^{34}$	$\mathbb{Z}^{10}$	$\mathbb{Z}^1$
Tübingen Triangle	$\mathbb{Z}^{24} \oplus \mathbb{Z}_5^2$	$\mathbb{Z}^5$	$\mathbb{Z}^1$
Socolar (undecorated)	$\mathbb{Z}^{28}$	$\mathbb{Z}^7$	$\mathbb{Z}^1$
Socolar (decorated)	$\mathbb{Z}^{59}$	$\mathbb{Z}^{12}$	$\mathbb{Z}^1$

Tiling (TTT) [BKSZ90, KSB93] by the triple $(V, \Gamma_{10}, \mathcal{W}_{10}^b)$, the undecorated octagonal Ammann-Beenker tiling [Ben82] by the triple $(V, \Gamma_8, \mathcal{W}_8^a)$, and the undecorated Socolar tiling [Soc89] by the triple $(V, \Gamma_{12}, \mathcal{W}_{12}^a)$. For the decorated versions of the Ammann-Beenker [Soc89, AGS92, G93] and Socolar tilings [Soc89], the set of singular lines $\mathcal{W}_n^a$ has to be replaced by $\mathcal{W}_n^a \cup \mathcal{W}_n^b$, $n = 8$ and 12, respectively. Finally, the heptagonal tiling in [GK00] is given by the triple $(V, \Gamma_{14}, \mathcal{W}_{14}^a)$. For all these tilings, the rational ranks given in [GK00] remain valid. The results of the 2 dimensional tilings are listed in Table 1.

The generalized Penrose tilings [PK87] are somewhat different from the tilings discussed above. They are built upon the decagonal lattice Γ_{10}, too, but have only fivefold rotational symmetry. The singular lines do not pass through the origin in general, and their positions depend on a continuous parameter γ. For instance, the representatives lines of the two translation orbits of lines parallel to e_0 pass through the points $-\gamma e_1$ and $\gamma(e_1 + e_2)$. It turns out that these shifts of line positions always lead to the same line intersections and incidences. Even multiple intersection points remain stable, and are only moved around if γ is varied. Consequently, all generalized Penrose tilings have the same homology, except for $\gamma \in \mathbb{Z}[\tau]$, which corresponds to the real Penrose tilings [deB81]. This had already been observed by Kalugin [Ka05], and is in contradiction with the results given in [GK00], which were obtained due to a wrong parametrisation of the singular line positions. Corrected results are given in Table 1.

Among the tilings discussed above, only the TTT has torsion in its homology. The set of singular lines of the TTT is constructed from the lines $\mathcal{W}_{10}^b$. The translation stabilizers Γ^α of all these lines are contained in a common sublattice Γ_{10}' generated by the star of vectors $e_i + e_{i+1}$; it has index 5 in Γ_{10}. It is therefore not too surprising that $\operatorname{coker}\beta_1$ (Theorem 4.1) develops a torsion component $\mathbb{Z}_5^2$, which shows up in the homology group H_0 of the TTT, in agreement with the results obtained by the AP method. In much the same way, a torsion

component $\mathbb{Z}_2$ in H_0 is obtained also for the octagonal tilings described by the data $(V, \Gamma_8, \mathcal{W}_8^b)$, but these tilings have not been considered before in the literature. The same applies also to the four-dimensional codimension 2 tilings with data $(V, \Gamma_{14}, \mathcal{W}_{14}^b)$, which have a torsion component $\mathbb{Z}_7^4$ in H_0, and $\mathbb{Z}_7^3$ in H_1, in agreement with the bounds given in Theorem 4.1.

There is an interesting relation of the TTT to the Penrose tiling. Since the lattice Γ'_{10} is rotated by $\pi/10$ with respect to Γ_{10}, the TTT can also be constructed from the triple $(V, \Gamma'_{10}, \mathcal{W}_{10}^a)$. However, the singular set $\Gamma'_{10} + \mathcal{W}_{10}^a$ is even invariant under all translations from Γ_{10}, so that it is equal to $\Gamma_{10} + \mathcal{W}_{10}^a$, which defines the Penrose tiling. In other words, the TTT and the Penrose tiling have the same set of singular lines, only the lattice Γ acting on it is different. The TTT is obtained by breaking the translation symmetry of the Penrose tiling to a sublattice of index 5. This explains why the Penrose tiling is locally derivable from the TTT, but local derivability does not hold in the opposite direction [BSJ91]. A broken symmetry can be restored in a local way, but the full lattice symmetry cannot be broken to a sublattice in any local way, because there are no local means to distinguish the five cosets of the sublattice. Tilings whose set of singular lines accidentally has a larger translation symmetry are likely candidates for having torsion in their homology.

6.4. **Codimension 3 examples with icosahedral symmetry.** There are four icosaheral tilings whose homology has been discussed in the literature so far: the Ammann-Kramer tiling [KN84], the Danzer tiling [Dan89], the canonical D_6 tiling [KP95], and the dual canonical D_6 tiling [KP95]. We give here an update to those results. In particular, we add the missing information on torsion, and also correct the rational ranks as outlined in Section 5. These examples also give a good overview of the different phenomena that can occur in the determination of torsion.

We start by describing the relevant lattices Γ and families of singular planes $\mathcal{W}$. In three dimensions, there are three inequivalent icosahedral lattices of minimal rank 6. The primitive lattice Γ_P is generated by a star of vectors pointing from the center to the vertices of a regular icosahedron. We choose any basis $e_1, \ldots, e_6$ from this vector star. The lattice Γ_F is then the sublattice of those integer linear combinations of the e_i, whose coefficients add up to an even integer. The lattice Γ_I is given by the $\mathbb{Z}$-span of the vectors in Γ_P, and the additional vector $\frac{1}{2}(e_1 + \ldots + e_6)$. These lattices are analogues of the the primitive, F-centered, and I-centered cubic lattices.[1] Γ_F is an index-2 sublattice of Γ_P, which in turn is an index-2 sublattice of Γ_I. The action of the icosahedral group A_5 on the three lattices gives rise to three integral representations, which are inequivalent under conjugation in $GL_6(\mathbb{Z})$. The singular

[1]strictly speaking, $\frac{1}{2}\Gamma_F$ is a centering of Γ_P

planes of all four examples have special orientations, being perpendicular either to a 5-fold, a 3-fold, or a 2-fold axis of the icosahedron (the latter are also parallel to a mirror plane). Moreover, each Γ-orbit of singular planes contains a representative which passes through the origin. We therefore define the families of planes $\mathcal{W}^n$, $n = 5, 3, 2$, consisting of all planes perpendicular to an n-fold axis, and passing through the origin. Denoting by V the internal space containing the lattices and singular planes, the Ammann-Kramer tiling is then defined by the triple $(V, \Gamma_P, \mathcal{W}^2)$, the dual canonical D_6 tiling by the triple $(V, \Gamma_F, \mathcal{W}^2)$, the Danzer tiling by the triple $(V, \Gamma_F, \mathcal{W}^5)$, and the canonical D_6 tiling by the triple $(V, \Gamma_F, \mathcal{W}^5 \cup \mathcal{W}^3)$. Interestingly, the sets $\Gamma_P + \mathcal{W}^2$ and $\Gamma_F + \mathcal{W}^2$ are invariant even under all translations from Γ_I, which means that they are both equal to $\Gamma_I + \mathcal{W}^2$. In other words, the sets of singular planes of the Ammann-Kramer tiling and the dual canonical D_6 tiling are the same, only the lattices acting on them by translation are different. Conversely, the sets of singular planes of the Danzer tiling and the canonical D_6 tiling have a lattice of translation symmetries which is equal to the lattice Γ_F they are constructed from. With these data, and the methods described in section 6.1, is it now straightforward to evaluate the formulæ (Theorem 5.1) for the ranks of the rational homology groups. The results are summarized in Table 2. As can be seen, compared to previously published results the rational ranks of H_0 and H_1 of the Ammann-Kramer tiling and the dual canonical D_6 tiling have been increased by 1, in agreement with Kalugin [Ka05], whereas all other rational ranks remain the same.

Next, we discuss the determination of torsion, which is possibly non-trivial only for H_0 and H_1. The torsion in H_1 is given by the torsion of $\mathrm{coker}\phi_2'$, which is straightforward to compute. The results are given in Table 2. For the torsion in H_0, however, we only have the exact sequence in Theorem 4.3. As ϕ_1' and ϕ_1'' are maps between free abelian groups, the torsions of $\mathrm{coker}\phi_1'$ and $\mathrm{coker}\phi_1''$ are not difficult to compute. The results are shown in Table 2. It is interesting to note that for all four examples considered, $\mathrm{coker}\phi_1''$ is not free, which will result in all four tilings having torsion in H_0. This is so even for a tiling as simple as the Danzer tiling, which has, like the Penrose tiling, only a single Γ-orbit of singular points. In fact, we do not know of any 3D icosahedral tiling having vanishing torsion in H_0. To determine the torsion in H_0, we further need the torsion of $\ker \phi_0'$, which is a map from $\oplus_{\alpha \in I_2} \mathrm{coker}\beta_1^\alpha$ to $\mathrm{coker}\gamma_1$, both of which are potentially non-free. In three of our four examples, $\mathrm{coker}\beta_1^\alpha$ is free for all $\alpha \in I_2$, hence also $\ker \phi_0'$ is free, and the torsion of H_0 is given by the direct sum of the torsions of $\mathrm{coker}\phi_1'$ and $\mathrm{coker}\phi_1''$. Our fourth example, the dual canonical D_6 tiling, is the only one which needs a more detailed analysis. In this case, $\mathrm{coker}\beta_1^\alpha$ is not free, but $\mathrm{coker}\gamma_1$ is, so that the torsion of $\ker \phi_0'$ can still be determined; in fact, $\mathrm{Torsion}(\ker \phi_0') = \mathbb{Z}_2^{15}$, because $\mathrm{Torsion}(\mathrm{coker}\beta_1^\alpha) = \mathbb{Z}_2$, and there are 15 planes α. We further have

TABLE 2. Homology of codimension 3 tilings with icosahedral symmetry, and data required for the determination of its torsion component. We use the abbreviations $t_1' = \text{Torsion}(\text{coker}\phi_1')$, $t_1'' = \text{Torsion}(\text{coker}\phi_1'')$, and $t_0' = \text{Torsion}(\ker \phi_0')$.

Tiling	t_1'	t_1''	t_0'	H_0	H_1	H_2	H_3
Ammann-Kramer	0	$\mathbb{Z}_2$	0	$\mathbb{Z}^{181} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{72} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{12}$	$\mathbb{Z}^1$
dual canonical D_6	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^7$	$\mathbb{Z}_2^{15}$	$\mathbb{Z}^{331} \oplus \mathbb{Z}_2^{26} \oplus \mathbb{Z}_4$	$\mathbb{Z}^{102} \oplus \mathbb{Z}_2^4 \oplus \mathbb{Z}_4$	$\mathbb{Z}^{12}$	$\mathbb{Z}^1$
Danzer	0	$\mathbb{Z}_2$	0	$\mathbb{Z}^{20} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{16}$	$\mathbb{Z}^7$	$\mathbb{Z}^1$
canonical D_6	0	$\mathbb{Z}_2$	0	$\mathbb{Z}^{205} \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^{72}$	$\mathbb{Z}^7$	$\mathbb{Z}^1$

$\text{Torsion}(\text{coker}\phi_1') = \mathbb{Z}_2^6$ and $\text{Torsion}(\text{coker}\phi_1'') = \mathbb{Z}_2^7$. As an additional piece of information, we can compute the rank T_0^2 of the torsion subgroup of $H_0(\Gamma, C^3)$, which by Proposition 3.1 is the difference of the ranks of $H_0(\Gamma; C^3 \otimes \mathbb{F}_2)$ and $H_0(\Gamma; C^3 \otimes \mathbb{Q})$ (note that 2 is the only prime p, for which T_0^p can be non-zero). We find

$$H_0(\Gamma; C^3 \otimes \mathbb{Q}) = \mathbb{Q}^{D_0}, \quad H_0(\Gamma; C^3 \otimes \mathbb{F}_2) = \mathbb{F}_2^{D_0+27},$$

so that $T_0^2 = 27$ ($D_0 = 331$).

Taking everything together, this means that the exact sequence (4.4) for $s = 0$ is of the form

$$0 \to \mathbb{Z}_2^{13} \oplus \mathbb{Z}^l \to T \oplus \mathbb{Z}^{D_0} \to \mathbb{Z}_2^{15} \oplus \mathbb{Z}^{D_0-l} \to 0$$

for some finite abelian group T with $T \otimes \mathbb{F}_2 = \mathbb{Z}_2^{27}$; moreover, it is immediate that T can contain only elements of order 2 and 4. Let us write $T = \mathbb{Z}_2^a \oplus \mathbb{Z}_4^b$ for non-negative integers a and b. As $T \otimes \mathbb{F}_2 = \mathbb{Z}_2^{27}$ we must have $a + b = 27$, but computing the order of T we also need $a + 2b = 28 - z$, where z is the rank of the cokernel of the inclusion $\mathbb{Z}^l \to \mathbb{Z}^{D_0}$. Together these say $b = 1 - z$, hence $b = 0$ or 1, and inspection shows the only two possible extensions are as follows. The first is

$$0 \to \mathbb{Z}_2^{12} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}^l \xrightarrow{\varphi} \mathbb{Z}_2^{26} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}^{D_0} \to \mathbb{Z}_2^{14} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}^{D_0-l} \to 0 \tag{6.1}$$

where $\varphi = \varphi_1 \oplus \varphi_2 \oplus \varphi_3$ respect the direct sums indicated, φ_1 and φ_3 are inclusions as direct summands, and $\varphi_2 : \mathbb{Z}_2 \to \mathbb{Z}_4$ is multiplication by 2. The second possible extension is

$$0 \to \mathbb{Z}_2^{13} \oplus \mathbb{Z} \oplus \mathbb{Z}^{l-1} \xrightarrow{\varphi'} \mathbb{Z}_2^{27} \oplus \mathbb{Z} \oplus \mathbb{Z}^{D_0-1} \to \mathbb{Z}_2^{14} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}^{D_0-l} \to 0 \tag{6.2}$$

where $\varphi' = \varphi_1' \oplus \varphi_2' \oplus \varphi_3'$ respects the direct sums indicated, φ_1' and φ_3' are inclusions as direct summands, and $\varphi_2' : \mathbb{Z} \to \mathbb{Z}$ is multiplication by 2. We note that the first extension corresponds to the case where one gets an exact sequence also of the torsion parts, that is, the right map of the last sequence

in Theorem 4.3 is onto. This could be used to distinguish the two cases, but we reason in another way.

The short exact sequence

$$0 \to \ker \phi_0 \to H_0(\Gamma; C^2) \xrightarrow{\phi_0} H_0(\Gamma; C^1_0) \to 0$$

upon tensoring with a commutative ring R gives (by standard homological algebra, for example see [HiSt]) an exact sequence

$$\cdots \to \mathrm{Tor}^1(H_0(\Gamma; C^1_0), R) \to \ker \phi_0 \otimes R \to H_0(\Gamma; C^2) \otimes R \xrightarrow{\phi_0} H_0(\Gamma; C^1_0) \otimes R \to 0\,.$$

However, by the universal coefficent theorem [Br], we also have $H_0(\Gamma; C^2 \otimes R) = H_0(\Gamma; C^2) \otimes R$ and $H_0(\Gamma; C^1_0 \otimes R) = H_0(\Gamma; C^1_0) \otimes R$. Moreover, since $H_0(\Gamma; C^1_0)$ is torsion free, we have $\mathrm{Tor}^1(H_0(\Gamma; C^1_0), R) = 0$. Thus we can make the identification

$$\ker_R \phi_0 = (\ker \phi_0) \otimes R$$

where $\ker_R \phi_0 := \ker\{\phi_0 : H_0(\Gamma; C^2 \otimes R) \to H_0(\Gamma; C^1_0 \otimes R)\}$. In general, however,

$$\mathrm{coker}_R \phi_1 \neq (\mathrm{coker} \phi_1) \otimes R$$

and so the above two extensions can be distinguished by comparing $\mathrm{coker}_{\mathbb{Z}_4} \phi_1$ with $(\mathrm{coker}\phi_1) \otimes \mathbb{Z}_4$. In fact, we deduce from the above that (6.1) implies

$$0 \to \mathrm{coker}_{\mathbb{Z}_4} \phi_1 \to \mathbb{Z}_2^{26} \oplus \mathbb{Z}_4^{D_0+1} \to \mathbb{Z}_2^{15} \oplus \mathbb{Z}_4^{D_0-l} \to 0 \tag{6.3}$$

and (6.2) implies

$$0 \to \mathrm{coker}_{\mathbb{Z}_4} \phi_1 \to \mathbb{Z}_2^{27} \oplus \mathbb{Z}_4^{D_0} \to \mathbb{Z}_2^{15} \oplus \mathbb{Z}^{D_0-l} \to 0. \tag{6.4}$$

In the first case, $\mathrm{coker}_{\mathbb{Z}_4} \phi_1$ has 2^{2l+13} elements and in the second case $\mathrm{coker}_{\mathbb{Z}_4} \phi_1$ has 2^{2l+12} elements. The number of elements of $\mathrm{coker}_{\mathbb{Z}_4} \phi_1''$ has been determined by computer; we find that there are 2^9 elements, and since $\mathrm{coker}_{\mathbb{Z}_4} \phi_1' = (\mathrm{coker} \phi_1') \otimes \mathbb{Z}_4$, as one easily checks, the first extension (6.1) is the right one. Hence

$$H_0(\Gamma; C^3) \cong \mathbb{Z}_2^{26} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}^{D_0}.$$

Summarizing, if $\ker \phi_0'$ is free, the torsion in H_0 can be determined relatively easily. Otherwise, an extension problem has to be solved. In the case of the dual canonical D_6 tiling, there were only two possible extensions, which were not too difficult to distinguish. It remains to be seen whether such an approach can still be successful also in more complicated cases.

7. General results

We conclude with two general results, one which limits the homological degrees in which torsion can occur, and the second which discusses the relation between the integral (co)homology of a canonical projection tiling and its K-theory.

7.1. **Bounds on where torsion can occur.** We consider a general canonical projection pattern of dimension d, codimension n, and assume the homology is finitely generated. By [FHK02b] homology will be non-zero only in dimensions $s = 0, \ldots, d$.

Theorem 7.1. *For a canonical projection pattern of dimension d, codimension n, and with finitely generated homology, there is no torsion in homology dimensions $s \geqslant \frac{(n-1)d}{n}$. Moreover, in homology dimensions $\frac{(n-2)d}{n} \leqslant s < \frac{(n-1)d}{n}$ the torsion is given by the torsion part of*

$$\operatorname{coker}\left\{ \bigoplus_{\alpha \in I_{n-1}} \Lambda_{n+s}\Gamma^\alpha \to \Lambda_{n+s}\Gamma \right\}$$

where α indexes the orbit classes of the $(n-1)$-dimensional singular hyperplanes in the internal space $\mathbb{R}^n$.

Proof. This is an extension of the results in dimensions $s \geqslant 1$ in Theorem 4.3 and may be proved in a similar fashion by induction on the codimension n, examining the top dimensions of the corresponding long exact sequences.

However, the most direct proof utilises the spectral sequence of [FHK02b]. Recall that the r^{th} column, $E^1_{r,*}$ for $0 \leqslant r \leqslant n$ is $H_*(\Gamma; C^r) = \bigoplus_{\Theta \in I_r} H_*(\Gamma^\Theta; C^r_\Theta)$ and, as noted in (2.4), these columns are only non-zero in respective degrees $0 \leqslant t \leqslant r\frac{d}{n}$. The differentials act $d^j \colon E^j_{r,s} \to E^j_{r-j,s+j-1}$.

First observe that for $s > \frac{(n-1)d}{n}$, the only non-zero entries in $E^1_{*,s}$ lie in the column $* = n$; thus these cannot be the source or target of any differential and as noted in Section 2 we read off

$$H_s(\Gamma; C^n) = E^1_{n,s} = E^\infty_{n,s} = H_{n+s}(\Gamma; \mathbb{Z}) = \mathbb{Z}^{\binom{n+d}{n+s}}$$

which of course is torsion free.

For the range of degrees, $\frac{(n-2)d}{n} < s \leqslant \frac{(n-1)d}{n}$, there are precisely two non-zero terms in $E^1_{*,s}$ and these are for $* = n$ and $n-1$. Here we have the possibility of only one non-trivial differential, namely d^1, and just two non-zero groups $E^\infty_{a,b}$ on the line $a + b = s$. Thus we obtain a long exact sequence

$$\cdots \to H_{s+1}(\Gamma; C^{n-1}) \xrightarrow{\psi_{s+1}} H_{n+s}(\Gamma; \mathbb{Z}) \to H_s(\Gamma; C^n) \xrightarrow{d^1} H_s(\Gamma; C^{n-1}) \xrightarrow{\psi_s} \cdots$$

and the ψ_s are interpreted in terms of the maps

$$\bigoplus_{\alpha \in I_{n-1}} \Lambda_{n+s-1}\Gamma^\alpha \to \Lambda_{n+s-1}\Gamma\,,$$

in direct analogy to the maps β_s of section 4.2 and the ϕ_s of 4.3. The result now follows as the only possibility for torsion in $H_s(\Gamma; C^n)$ must arise from $\operatorname{coker}(\psi_{s+1})$. Note that for $s = (n-1)d/n$ this cokernel is torsion free since

the source of $\psi_{(n-1)\frac{d}{n}+1}$ is zero. The result for $s = \frac{(n-2)d}{n}$ is similar but more involved as there can also be a non-trivial differential $d^1 \colon E^1_{n-1,s} \to E^1_{n-2,s}$ for this s. $\square$

Corollary 7.2. *For a system as in Theorem 7.1,*

$$H_s(\Gamma; C^n) = \mathbb{Z}^{\binom{n+d}{d-s}} \quad \textit{for} \quad s > \frac{(n-1)d}{n}.$$

$\square$

7.2. ***K*-theory.** Our most straightforward results deal with the case of tilings in dimensions $d \leqslant 3$.

Theorem 7.3. *For canonical projection patterns in $\mathbb{R}^d$ with $d \leqslant 3$ and finitely generated homology there are isomorphisms*

$$K^0(MP) \cong \bigoplus_r H^{2r}(\Gamma; C^n) \qquad K^1(MP) \cong \bigoplus_r H^{2r+1}(\Gamma; C^n)\,.$$

Proof. We have noted before that for finitely generated homology, n must divide d, so for $d = 3$ we must have $n = 1$ or 3, and for $d = 2$ we must have $n = 1$ or 2. If $n = 1$ we have seen the computations are essentially those for the homology of a punctured torus and it is immediate that the K-theory and the cohomology will agree.

For the larger codimension cases, we consider the Atiyah-Hirzebruch type spectral sequence linking $H^*(\Gamma; C^n)$ with the K-theory $K^*(MP)$. This has an E_2-term

$$A_2^{r,s} = \begin{cases} H^r(\Gamma; C^n) = H_{d-r}(\Gamma; C^n) & \text{for } s \text{ even,} \\ 0 & \text{for } s \text{ odd,} \end{cases}$$

and differentials $d_j \colon A_j^{r,s} \to A_j^{r+j,s-j+1}$. Of course $A_2^{r,s} = 0$ if $r > d$.

Thus, if $d = 2$ there can be no non-trivial differentials as there is no appropriate pair of non-zero entries between which a differential can act. For $d = n = 3$ there is at first sight the possibility of a non-trivial differential

$$d_3 \colon \mathbb{Z} = A_2^{0,2} \to A_2^{3,0} = H_0(\Gamma; C^3),$$

but in fact this cannot be non-zero: by comparison with the Atiyah-Hirzebruch type spectral sequence for a point we see that the column $A_*^{0,*}$ can support no non-trivial differentials. Thus in all cases $A_2^{*,*} = A_\infty^{*,*}$.

It remains to check that there are no extension problems. In the case $d = n = 3$ we have extensions

$$\begin{array}{ccccccccc} 0 & \to & H^2(\Gamma; C^3) & \to & K^0(MP) & \to & H^0(\Gamma; C^3) & \to & 0 \\ 0 & \to & H^3(\Gamma; C^3) & \to & K^1(MP) & \to & H^1(\Gamma; C^3) & \to & 0 \end{array} \tag{7.1}$$

The quotient groups in each exact sequence are $H^0(\Gamma; C^3) = H_3(\Gamma; C^3) = \mathbb{Z}$ and $H^1(\Gamma; C^3) = H_2(\Gamma; C^3)$, which by Section 4 are torsion free and, by assumption, finitely generated. Thus they are free abelian and so the exact sequences (7.1) split as claimed. The case $d = n = 2$ is similar, though simpler, as the exact sequence for $K^1(MP)$ directly gives an isomorphism $K^1(MP) = H^1(\Gamma; C^2)$. □

This, together with the calculations of Section 6, yields the following, showing that there are examples of torsion in the K-theory of an aperiodic tiling. The same result holds for the K-theory of the non-commutative C^*-algebras associated to this tiling, as considered in [FHK02a, FHK02b].

Corollary 7.4. *For the Tübingen Triangle Tiling, $K^0(MP) = \mathbb{Z}^{25} \oplus \mathbb{Z}_5^2$ and $K^1(MP) = \mathbb{Z}^5$.* □

Theorem 7.3 admits a number of possible generalisations in particular cases. A classical result tells that the differentials in the Atiyah-Hirzebruch spectral sequence are always torsion valued, and moreover the first differential that can non-trivially take a p-torsion value is d_{2p-1}. Thus, for example, if $H_*(\Gamma; C^n)$ is torsion free, this spectral sequence will collapse and an assumption of finitely generated homology will mean that all extension problems are trivial, yielding the following result, originally claimed in [FHK02a, FHK02b].

Theorem 7.5. *For canonical projection tilings with finitely generated, torsion free homology there is an isomorphism*

$$K^*(MP) \cong \bigoplus H^*(\Gamma; C^n)\,.$$

□

More subtly, if the lowest primary torsion in $H_*(\Gamma; C^n)$ is p and $d \leqslant 2p - 1$, then again we can deduce that the spectral sequence collapses. Depending on which cohomological degrees the torsion occurs in, there may or may not be extension problems.

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References

[AGS92] R. Ammann, B. Grünbaum, and G.C. Shephard, Aperiodic Tiles, *Discrete Comput. Geom.* 8 (1992) 1–25.

[AP98] J.E. Anderson and I.F. Putnam, Topological invariants for substitution tilings and their associated C^*-algebras, *Ergod. Th. & Dynam. Syst.* 18 (1998) 509–537.

[BKSZ90] M. Baake, P. Kramer, M. Schlottmann, and D. Zeidler, Planar patterns with fivefold symmetry as sections of periodic structure in 4-space, *Int. J. Mod. Phys. B* 4 (1990) 2217–2268.

[BSJ91] M. Baake, M. Schlottmann, P.D. Jarvis. Quasiperiodic tilings with tenfold symmetry and equivalence with respect to local derivability *J. Phys. A: Math. Gen.* 24 (1991) 4637–4654.

[Ben82] F.P.M. Beenker, Eindhoven University of Technology Report 82-WSK-04 (1982).

[Br] K. Brown, Cohomology of Groups, Springer 1982.

[deB81] N.G. de Bruijn, Algebraic theory of Penrose's aperiodic tilings of the plane, *Math. Proc.* A84 (1981) 39-66.

[Coh95] H. Cohen, A course in computational algebraic number theory, 2nd edition, Springer, Berlin, Heidelberg 1995.

[Dan89] L. Danzer, Three dimensional analogues of the planar Penrose tilings and quasicrystals, *Discrete Math.* 76 (1989) 1-7.

[EGN97] B. Eick, F. Gähler, W. Nickel. Computing Maximal Subgroups and Wyckoff Positions of Space Groups *Acta Cryst.* A53 (1997) 467–474.

[Cryst] B. Eick, F. Gähler, W. Nickel. Cryst – Computing with crystallographic groups, Version 4.1.4 (2004)) (`http://www.gap-system.org/Packages/cryst.html`).

[FH99] A.H. Forrest and J.R. Hunton. The cohomology and K-theory of commuting homeomorphisms of the Cantor set. *Ergod. Th. and Dynam. Sys.*, 19 (1999) 611-625.

[FHK01] A.H. Forrest, J.R. Hunton and J. Kellendonk. Topological invariants for projection method patterns. *XIIIth International Congress of Mathematical Physics*, London 2000 Proceedings, pp333-339, Int. Press, Boston, MA, 2001, Ed. A. Fokas, A. Grigoryan, T. Kibble and B. Zegarlinski. ISBN 1-57146-085-3.

[FHK02a] A.H. Forrest, J.R. Hunton and J. Kellendonk. Cohomology of canonical projection tilings. *Comm. Math. Phys.* 226 (2002) 289-322.

[FHK02b] A.H. Forrest, J.R. Hunton and J. Kellendonk. Topological invariants for projection method patterns. *Mem. Amer. Math. Soc.* 758 (2002).

[G04] F. Gähler, Lecture given at PIMS Workshop on *Aperiodic Order: Dynamical Systems, Combinatorics, and Operators*, Banff, June 2004.

[G93] F. Gähler. Matching Rules for Quasicrystals: The Composition-Decomposition Method *J. Non-Cryst. Solids* 153 & 154 (1993) 160–164.

[GK00] F. Gähler and J. Kellendonk. Cohomology groups for projection tilings of codimension 2. *Materials Science and Engineering.* 438-440 (2000) 294-296.

[GAP] The GAP Group GAP – Groups, Algorithms, and Programming, Version 4.4 (2004) (`http://www.gap-system.org`).

[HiSt] P. J. Hilton and U. Stammbach. A course in homological algebra. Springer GTM 4 (1971).

[Ka05] P. Kalugin. Cohomology of quasiperiodic patterns and matching rules, J. Phys. A: Math. Gen. 38 No 14 (2005) 3115-3132.

[K97] J. Kellendonk. The local structure of tilings and their integer group of coinvariants. *Commun. Math. Phys.*, 187(1):115–157, 1997.

[KP95] P. Kramer and Z. Papadopolos, *Models of icosahedral quasicrystals from 6d lattices*, Proceedings of the International conference on Aperiodic Crystals, Aperiodic '94 (G. Chapuis et al., ed.), World Scientific, Singapore, 1995, pp. 70–76.

[KSB93] R. Klitzing, M. Schlottmann, and M. Baake, Perfect matching rules for undecorated triangular tilings with 10-, 12-, and 8-fold symmtery, *Int. J. Mod. Phys. B* 7 (1993) 1455–1473.

[KN84] P. Kramer and R. Neri, On periodic and non-periodic space fillings of $\mathcal{E}^m$ obtained by projection, *Acta Cryst A* 40 (1984) 580–587.

[Mc01] J. McCleary, A user's guide to spectral sequences, 2nd edition, Cambridge Studies in Advanced Mathematics 58, Cambridge University Press 2001.

[PK87] A. Pavlovitch and M. Kleman, Generalised 2D Penrose tilings: structural properties, *J. Phys. A: Math. Gen.* 20 (1987) 687–702.

[Soc89] J.E.S. Socolar. Simple octagonal and dodecagonal quasicrystals *Phys. Rev. B 39* (1989) 10519–10551.

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